Effects of network topology, transmission delays, and refractoriness on the response of coupled excitable systems to a stochastic stimulus

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In a recent report,[2] we presented an analysis of the

B. Model dynamics

By considering a large ensemble of realizations of the above stochastic process on the same network, we can define the probability that node i is at state x_i^t at time t as $p_i^t(x)$. The probabilities p_i^t evolve in one time step by

$$p_i^{t+1}(1) = p_i^t(0) r_i^t, \tag{2}$$

$$p_i^{t+1}(2) = p_i^t(1), \tag{3}$$

$$\cdots \tag{4}$$

$$p_i^{t+1}(m_i) = p_i^t(m_i - 1), \tag{5}$$

and we also have the normalization condition

$$p_i^t(0) = 1 - \sum_{j=1}^{m_i} p_i^t(j), \tag{6}$$

where r_i^t in Eq. (2) is the rate of transitions from the ready to the excited state, given by

$$r_i^t = E\left[\eta + (1 - \eta)\left\{ 1 - \prod_j \left(1 - A_{ij} I_j^{t-\tau_{ij}}\right) \right\} \right], \tag{7}$$

where I_j^t is one if node j is excited at time t and zero otherwise, and $E[\cdot]$ denotes an ensemble average. Assuming that the neighbors of node i being excited are independent events, we obtain, letting $p_i^t(1) \equiv p_i^t$,

$$r_i^t =$$

Now, we use the fact that the largest eigenvalue of A, λ, is determines $\bar{p}_i$) and decays exponentially with the number of expected excitations from its neighbors. In terms of the aggregate response $\hat{F}$, Eq. (22) becomes, after multiplying by A_{ki}, summing over k and i, and normalizing,

$$\frac{d\hat{F}}{d\eta} = \frac{\langle d^{out}\bar{p}^2 e^{-A\bar{p}}\rangle}{\langle d\rangle}. \tag{23}$$

C. Dynamics near the critical regime

More precisely, the value of stimulus η_{low} (η_{high}) corresponding to a low (high) threshold of activity $\hat{F}_{\mathrm{low}}$ ($\hat{F}_{\mathrm{high}}$) are found and the dynamic range is calculated as

$$\Delta = 10 \log_{10}(\eta_{\mathrm{high}}/\eta_{\mathrm{low}}). \tag{31}$$

Using our approximations to the response $\hat{F}$ as a function of stimulus η

and then connecting them using the configuration model.[29] In some cases, an additional fourth step was used to change the assortativity coefficient ρ, defined in Eq. (30), of a critical (i.e., with $\lambda = 1$) scale-free network, making this network more assortative (disassortative) by choosing two links at random, and swapping their destination connections only if the resulting swap would increase (decrease) ρ. This swapping allows for the degree of assortativity (and thereby, λ) to be modified while preserving the network's degree distribution.[8,19]

B. Results of numerical experiments

We first demonstrate the ability of the non-perturbative approximation to predict aggregate network behavior in a variety of conditions. Fig. 2 shows a multitude of simulations (symbols) with the predicted behavior of Eq. (17) overlaid (lines). The cases considered in Fig. 2 include different combinations of topology, assortativity, largest eigenvalue λ, delays, and number of refractory states. The number of refractory states m_i was chosen either constant, $m_i = m$, or randomly chosen with equal probability among $\{1, 2, \ldots, m_{max}\}$

of the refractory states. Equation (21) predicts that the response should scale as $\langle m + 1/2 \rangle^{-1}$. The inset shows how, after multiplication by $\langle m + 1/2 \rangle$, the response curves collapse into a single curve. Figure 4 also depicts a linear relationship, $\hat{F} \sim (\lambda - 1)$ for $\lambda > 1$. Making a connection with the theory of nonequilibrium phase transitions in which $\hat{F} \sim (\lambda - \lambda_c)^{\beta}$, we derive $\lambda_c = 1$ and the critical exponent $\beta = 1$.

Figure 5 shows the response $\hat{F}$ close to $\eta = 1$ calculated for various values of m from the simulation (symbols), and from Eq. (23) (solid lines). Equation (23) describes well the slope of $\hat{F}$ close to $\eta = 1$. An important observation is that as m grows, the relative slope $\hat{F}^{-1} d\hat{F}/d\eta$ at $\eta = 1$ decreases. Therefore, if the typical refractory period m is large, the response $\hat{F}$ saturates [e.g., reaching 90% of $\hat{F}(1)$] for smaller values of η.

Transmission delays, as in the analogous system of gene regulatory networks,[18] determined from the distribution of delays. In Fig. 6, we