Synchronization[1–7] pervades physical and biological systems[8,9]. It is key to characterize physiological[10] and brain rhythms[11], to understand collective animal behavior[12], and is also observed in non biological systems such as coupled Josephson junctions[13], lasers[14] involved in the appearance of brain rhythms and cortical oscillations[11,66,83] since Kuramoto-like dynamics has been

However, the adaptive coupling proposed in[26] is not local: it does not admit a generalization that locally couples the different associated to the links of the network, giving

$$
\begin{aligned}
\dot{\bullet} &= \quad -\frac{}{N} B \sin \; B^{\top} \bullet + K_{[1]}^{-1} L_{[1]} \quad , \\
\dot{} &= \; \tilde{} - \frac{}{N} B^{\top} \sin \; B \;\; - K_{[0]}^{-1} L_{[0]} \bullet \;\; .
\end{aligned} \tag{15}
$$

observed at $\;^{\star}_{c}(N) < 2.14623\ldots$ in finite networks. We study this effect quantitatively by measuring the transition threshold on systems of varying size N, averaging 100 independent iterations for each N (see Fig. 4). We find that the observed distance from the theoretical critical point given in Eq. (31) decreases consistently with a power-law in N, with scaling exponent 0.177 (computed with integration time $T_{max} = 5$), confirming further our theoretical prediction of $\;^{\star}_{c}$. The observed behavior is consistent with the earlier transition being caused by finite N effects. For finite N, the system will have fluctuations about the incoherent state which may bring the system to the basin of attraction of the rhythmic phase and cause a transition even when the incoherent state is stable. Thus, the observed transition point depends on N (larger N implies smaller fluctuations), and T_{max} (larger T_{max} implies larger probability of transition before reaching $\;^{*}_{c}$).

Finally, we find that &

quantity. The expansion coefficients are given by

$$
\begin{aligned}
f_n^{(,i)}(\ , \hat{} , t) &= [a_i(\ , \hat{} , t)]^n \\
f_m^{(,i)}(\ , \hat{} , t) &= [b_i(\ , \hat{} , t)]^m
\end{aligned} \tag{37}
$$

for $n > 0, m > 0$. The series in Eq. (36) converges for $|a_i| \leq 1$ and $|b_i| \leq 1$ provided that we attribute to and an infinitesimally small imaginary part.

For $X \neq 0$, $a_i \neq 0$ and $b_i \neq 0$, the continuity equation is satisfied if and only if (see Methods for details) a_i and b_i are complex variables with absolute value one, i.e., $|a_i| = |b_i| = 1$, that satisfy the system of differential equations

$$
\begin{aligned}
\partial_t a_i + i a_i \ _{,i} + \tfrac{1}{2} \ X \ a_i^2 - X^\star \ - \ \tfrac{1}{4} a_i (b_i - b_i^{-1}) &= 0, \\
\partial_t b_i + i b_i \ _{,i} + \tfrac{1}{2} \ \bar{c} \ X \ a_i - X^\star a_i^{-1} \ b_i + \ \tfrac{1}{2} (b_i^2 - 1) &= 0,
\end{aligned} \tag{38}
$$

where here and in the following we indicate with $X^\star$ the complex conjugate of X, and with $a_i^{-1} = a_i^\star$ and $b_i^{-1} = b_i^\star$ the complex conjugate of a_i and b_i respectively. We note that the only stationary solutions of these equations are

$$
\begin{aligned}
a_i &= \ -i d_{i,} \ \pm \sqrt{1 - d_{i,}^2} \\
b_i &= \ -i d_{i,} \ \pm \sqrt{1 - d_{i,}^2} \ ,
\end{aligned} \tag{39}
$$

with $d_{i,}$, $d_{i,}$ defined as

$$
\begin{aligned}
d_{i,} \ &= \frac{\ _i - \ \hat{}}{R}, \\
d_{i,} \ &= -\hat{} \ _i / \ + \mathrm{Im} X \ ,
\end{aligned} \tag{40}
$$

and having absolute value $|d_{i,} \ | \leq 1$ and $|d_{i,}$

and $b_i(\ _i, \hat{\ }_i)$ as

$$
\begin{aligned}
X &= \frac{1}{N}\sum_{i=1}^{N} a_i^\star(\ _i, \hat{\ }_i), \\
X &= \frac{1}{N}\sum_{i=1}^{N} b_i^\star(\ _i, \hat{\ }_i),
\end{aligned}
\tag{45}
$$

where $a_i^\star$ and $b_i^\star$ are the complex conjugates of a_i and b_i, respectively.

If the internal frequencies $\ _i$ and $\hat{\ }_i$ are not known, we can express these complex order parameters in terms of the marginal distributions $G_0(\ )$ and $G_1(\hat{\ })$ as

$$
\begin{aligned}
X &= \int d \ \int d\hat{\ }\, G_0(\ )G_1(\hat{\ })a^\star(\ , \hat{\ }), \\
X &= \int d \ \int d\hat{\ }\, G_0(\ )G_1(\hat{\ })b^\star(\ , \hat{\ }).
\end{aligned}
\tag{46}
$$

This derivation shows that $a(\ , \hat{\ })$ and $b(\ , \hat{\ })$ can be obtained from the integration of (Eqs. (38) and (41)). In particular, as we discuss in the next paragraph (paragraph II E 2) these equations will be used to investigate the steady state solution of this dynamics and the range of frequencies on which this stationary solution can be observed. However for Dirac synchronization we observe a phenomenon that does not have an equivalent in the standard Kuramoto model. Indeed Eqs. (38) and (41) do not always admit a coherent stationary solution, and actually the non-phase as a rhythmic phase with

$$(t) \simeq \quad (0) + \quad {}_E t, \tag{53}$$

where we note, however, that numerical simulations reveal that the observed emergent frequency $\ _E$ decreases with increasing network size (see Fig. 8).

consistent equation for R as

$$1 = \int_{-1}^{1} dx G_0(\;_0 + \; R\, x)\sqrt{1-x^2} \int_{-} d\hat{\;} G_1(\hat{\;})$$
$$+ \int_{|\hat{\;}| \geq} d\hat{\;} G_1(\hat{\;}) \int_{-1}^{1} dx G_0(\;_0 + \; R\, x - \hat{\;}/2)\sqrt{1-x^2}. \quad (70)$$

For R $\ll 1$, we make the following approximations

$$G_0(\;_0 + xR\, x) \simeq G_0(\;_0),$$
$$G_0(\;_0 + \; R\, x - \hat{\;}/2) \simeq G_0(\;_0 - \hat{\;}/2). \quad (71)$$

Inserting these expressions into the self-consistent equation for R , we can derive the equation determining the value of the coupling constant $\; = \;_c$ at which we observe the continuous phase transition,

$$1 = \sqrt{\frac{}{2}}\left[\frac{1}{2}\,\mathrm{erf}\;\frac{}{\sqrt{2}} + \frac{1}{\sqrt{5}}\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{5}{2}}\;\right)\right] \quad (72)$$

where erf(x) is the error function and erfc(x

Moreover, for every $m > 0$, $n > 0$ the following equation needs to be satisfied

$$na^{n-1}b^m\partial_t a + ma^n b^{m-1}\partial_t b + ia^n b^m n \quad + ia^n$$

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