Downlink Performance Analysis for a Generalized Shotgun Cellular System

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Abstract **In this paper, e anal e the signal-to-interference-pl s-noise ratio (SINR) performance at a mobile station (MS) in**

LANs in an apartment building (note that to model urban areas the BS density function might need to be heterogeneous). The 1-D, 2-D, and 3-D SCSs are described using the BS density functions $d(x)$, $d(r, \ )$, and $d(r, \ , \phi)$, where $-\infty \leq x \leq \infty$ represents a point in 1-D, and $r, \ , \phi$ are used to represent a point in polar coordinates, in 2-D and 3-D.

A l-D SCS is said to be homogeneous if the BS density function is a constant over the entire l-D space. A homogeneous 2-D SCS is a common model for the random node placement in many scenarios.

We consider the most general possible description for the wireless radio environment. Let the received power at a distance $r(\geq 0)$ from a given BS be given by

$$P(r) = K\psi/h(r), \tag{1}$$

where K represents the transmission power and the antenna gain of the BS, ψ captures the channel fading, and the function $h(\cdot)$ represents a path-loss[1]

where (a) is obtained by exchanging the order of integration and differentiation, which is valid since $\lambda(r)$ is continuous. Further, the resultant integral can be written in terms of the Laplace transform of $\lambda(x)$. Using $\mathcal{L}\ \lambda(x), \frac{1}{\ }\ _{=0} = 0$ as the initial condition, the above differential equation can be solved to obtain the condition for equivalence between the two SCSs to be (8). ■

The following shows examples for the existence of BS density functions $(\lambda(r), \lambda(r))$ that satisfy the condition in (8).

1 Polynomial—polynomial equivalence: The pair $(\lambda(r), \lambda(r)) = (\ _1 r\ ,\ \ _2 r\ )$ satisfy the condition in (8) as long as $\ +1 > 0$, and $\ _1 = \ _2\ (1+\ ) > 0$, where $(\cdot)$ is the Gamma function.

2 Rational—exponential equivalence: The pair $(\lambda(r), \lambda(r)) = \frac{1}{(1+\ )^2}, -\ \ , \forall\ > 0$ satisfy the condition in (8).

We will see in the following section that the equivalent 1-D BS density function for the homogeneous l-D SCSs are polynomial functions, and using Example 1 and Lemma 2, simple analytical expressions for the tail probability of SINR are obtained.

The results presented in this section can together accurately characterize the SINR in any arbitrary SCS with arbitrary transmission and channel characteristics. The semi-analytical expressions presented above might seem unwieldy at the first glance. But it turns out that several insightful results can be extracted from this representation for a special class of SCSs that are practically important and popular in literature. This special class of SCSs are the homogeneous l-D SCSs, $l \in \{1, 2, 3\}$, and we dedicate the next section to studying this special class in detail.

IV. Homogeneous l-D SCS

In this section, we focus on the analysis of the homogeneous l-D SCSs with a power-law path-loss model $h(R) = R^{\varepsilon}$. The homogeneous l-D SCS is the most widely used stochastic geometric model in the literature for modeling arrangement of node locations. Especially, its validity in the study of the small-cell networks is extremely appealing. Moreover, this model has the advantage of being analytically amenable for a variety of situations that are of great importance in the modeling and analysis of any type of wireless network. The results provide several insights about such large-scale networks that can be applied in the design of actual networks in practice. Next, we apply the results of the previous section to the case of the homogeneous l-D SCS.

1. *1* A homogeneous l-D SCS with a constant BS density λ_0 over the entire space is equivalent to the 1-D SCS with a BS density function $\lambda(r) = \lambda_0 b\ r^{\ -1}, \forall r \geq 0$, where $b_1 = 2$, $b_2 = 2\pi$, $b_3 = 4\pi$.

This is easily proved by letting $d(x)$, $d(r,\ )$, and $d(r,\ ,\phi)$ be λ_0 in Proposition 1.

For the power-law path-loss model $1/h(R) = R^{\varepsilon}$, we have the following equivalent SCS using Corollary 1 and Theorem 2.

2. *2* A homogeneous l-D SCS with BS density λ_0 and path-loss model $\frac{1}{\varepsilon}$ is equivalent to the 1-D SCS with a BS density function $\lambda(r) = \lambda_0 \frac{l}{\varepsilon} r^{\frac{l}{\varepsilon}-1}$, $r \geq 0$ and the path-loss model $\frac{1}{\ }$.

Next, we characterize the effect of random transmission powers and fading factors, i.i.d. across BSs in the homogeneous l-D SCS. The effect of fading factors with arbitrary distribution on the SINR of homogeneous 2-D SCS has been reported in [18], [29], [30], and the following result generalizes it further.

2 A homogeneous l-D SCS with BS density λ_0, power-law path-loss model $\frac{1}{\varepsilon}$, random transmission powers and fading factors that have arbitrary joint distribution and are i.i.d. across all the BSs is equivalent to another homogeneous l-D SCS with a BS density $\lambda = \lambda_0 \mathbb{E}\ (K\ )^{\frac{l}{\varepsilon}}$, same power-law path-loss model $\frac{1}{\varepsilon}$, unity transmission power and unity fading factor at each BS, where K, have the same joint distribution as the transmission power and fading factors of the original homogeneous l-D SCS and $\mathbb{E}[\cdot]$ is the expectation operator w.r.t. K and , as long as $\mathbb{E}\ (K\ )^{\frac{l}{\varepsilon}} < \infty$.

Using Corollary 1 and Corollary 2, we obtain a 1-D SCS with BS density function $\lambda(r) = \lambda_0 \frac{l}{\varepsilon} r^{\frac{l}{\varepsilon}-1}$, with a path-loss model $\frac{1}{\ }$. Now, from Theorem 2, the equivalent canonical SCS has a BS density function $\hat{\lambda}(r) = \mathbb{E}\ (K\ )^{\frac{l}{\varepsilon}} \times \lambda(r)$. This result can be traced back to the scaling of the BS density of the original homogeneous l-D SCS by $\mathbb{E}\ (K\ )^{\frac{l}{\varepsilon}}$. ■

As a result, we can restrict our attention to SINR characterization when all the BSs of the l-D SCS have unity transmission power and fading factors. Now, we give the expression for the tail probability of SINR in a homogeneous l-D SCS.

4. In a homogeneous l-D SCS with a BS density λ_0, unity transmission power and fading factor at each BS, if the path-loss exponent of the power-law path-loss model satisfies $\ > l$, the characteristic function of the reciprocal of SINR is given by

$$\ _{\frac{1}{\mathrm{SINR}}}(\ ) = \mathrm{E}\ _1\ \ \omega\ \ ^1 \times \ \frac{\lambda_0 b_l}{l}\ _1^{\frac{l}{\varepsilon}} \left(1 - {}_1F_1\left(-\tfrac{l}{\varepsilon}; 1-\tfrac{l}{\varepsilon};\ \omega\right)\right)\ , \tag{9}$$

where the p.d.f. of R_1 is $f\ _1(r) = \lambda_0 \frac{l}{\varepsilon} r^{\frac{l}{\varepsilon}-1} \cdot\ ^{-\ _0 \frac{b_l}{l}\ \frac{l}{\varepsilon}}$, $r \geq 0$. When $\ = 0$, the SINR is equivalently the signal-to-interference ratio (SIR), and

$$\ _{\frac{1}{\mathrm{SIR}}}(\ ) = \frac{1}{{}_1F_1\ -\frac{\ }{\varepsilon}; 1-\frac{\ }{\varepsilon}; i\ }, \tag{10}$$

where ${}_1F_1(\ldots)$ is the confluent hypergeometric function of the first kind [31]. The tail probability of SINR is given by (4).

From Corollary 2, the SINR distribution is equivalent to the canonical SCS with BS density function $\lambda(r) = \lambda_0 \frac{l}{\varepsilon} r^{\frac{l}{\varepsilon}-1}$, $r \geq 0$. Now, by solving for (6), in Theorem 3, we obtain (9). Further, the expectation in (9) reduces to (10). ■

Due to Corollary 3, the homogeneous l-D SCS satisfies the conditions in Lemma 2 and hence a simple expression for the tail probability of SINR for $\ \geq 1$ can be derived. A special case of the following result for the homogeneous 2-D SCS and exponential fading case was reported in [32].

2 For a homogeneous l-D SCS with BS density λ_0, path-loss model $\frac{1}{\varepsilon}$, $> l$, with unity transmission power and fading factor at each BS, the tail probability of SINR for ≥ 1 is

$$\mathbb{P}(\{\mathrm{SINR} > \ \}) = \int_{=0}^{\infty} \frac{\lambda_0 b\, r^{-1}}{1 + \overline{\varepsilon}} \quad - r^{\varepsilon} - \frac{\lambda_0 b\, r\, \pi^{\frac{l}{\varepsilon}}}{1 + \overline{\varepsilon}} \quad \overline{\varepsilon} \, dr, \quad (11)$$

$$= {}^{-\frac{l}{\varepsilon}} \mathbb{P}(\{\mathrm{SINR} > 1\}), \quad (12)$$

and when $= 0$, the tail probability of SIR is

$$\mathbb{P}(\{\mathrm{SIR} > \ \}) = \frac{\overline{\varepsilon}^{\,-\frac{l}{\varepsilon}}}{\overline{\varepsilon}} = \ l\, \overline{\ } \ {}^{-\frac{l}{\varepsilon}}. \quad (13)$$

Due to Corollary 3, the homogeneous l-D SCS is equivalent to another homogeneous l-D SCS with the same path-loss model and transmission powers as the former, and with a BS density $\frac{0}{\left(1+\frac{l}{\varepsilon}\right)}$ and i.i.d. unity mean exponential

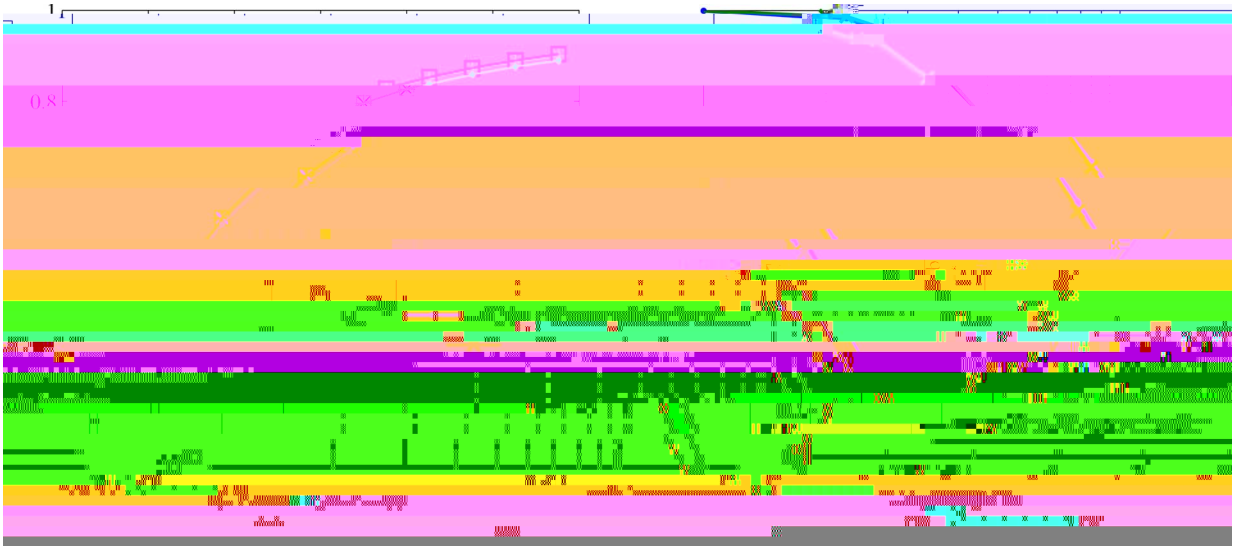

Fig. 3. (a) Comparison of Simulations with the analytical results for a homogeneous 2-D SCS. (b) Comparing exact SIR and the few BS approximation for path-loss $\varepsilon = 4$.

and the pluses $(+)$ represent the values computed analytically and by Monte-Carlo simulations, respectively. According to Remark 3, the same figure can be used for 1-D and 3-D systems with path-loss exponent $'$ using the scaling $= 2\,'$ and $= \frac{2}{3}\,'$.

In the following, we present an approximation to SIR based on modeling the interference due to the strongest few BSs accurately and the interference due to the rest by their ensemble average. The approximation is expected to be tight for low BS densities. Due to Remark 1, the same approximation will be tight for all BS densities. Now, we define the so-called and derive closed form expressions for the tail probability of SIR at MS in a , l-D SCS for both the SIR regions [0,1) and $[1, \infty)$.

4 The few BS approximation corresponds to modeling the total interference power at the MS in a SCS as the sum of the contributions from the strongest few interfering BSs and an ensemble average of the contributions of the rest of the interfering BSs.

Recall that the total interference power is $P_I = \sum^{\infty}_{=2} R^{-\varepsilon}$, where $\{R\}^{\infty}_{=1}$ is the set of distances of BSs arranged in the ascending order of their separation from the MS. The arrangement also corresponds to the descending order of their contribution to P_I, due to path-loss. In the few BS approximation, P_I is approximated by $P_I(k) = \sum_{=2} R^{-\varepsilon} + \mathbb{E}[\sum^{\infty}_{=+1} R^{-\varepsilon} | R]$, for some k, where $\mathbb{E}[\cdot]$ is the expectation operator and corresponds to the ensemble average of the contributions of BSs beyond R. The SIR at the MS obtained by the few BS approximation is denoted by SIR. The expectation is calculated as follows.

For a homogeneous l-D SCS, with BS density λ_0 and $> l$, for $k = 1, 2, 3, \cdots$,

$$\mathbb{E}\left[\sum_{=+1} R^{-\varepsilon} \,\Big|\, R\right] = \frac{\lambda_0 b\, R^{-\varepsilon}}{-l}. \tag{15}$$

Firstly, use Corollary 1 to reduce the l-D SCS to an equivalent 1-D SCS with BS density function $\lambda(r) = \lambda_0 b\, r^{-1}$, $\forall r \geq 0$. Next, given k, using the Superposition theorem of Poisson processes, the original Poisson process is equivalent to the union of two independent Poisson processes defined in the non-overlapping regions $[0, R]$ and (R, ∞), respectively, with the same BS density function. Now, using Campbell's theorem [1, (3.18), p. 28] to the Poisson process defined in (R, ∞), we obtain (15). ■

The following theorem gives the SIR tail probability approximation, using $k = 2$.

4 In a homogeneous l-D SCS with BS density λ_0 and path-loss exponent , satisfying $> l$, the tail probability of SIR_2 at the MS is given by

$$\mathbb{P}(\{\mathrm{SIR}_2 > \}) = \begin{cases} {}^{-\frac{l}{\varepsilon}} C_{\frac{\varepsilon}{l}}, & \geq 1 \\ 1 - e^{-()}(1 + u()) + {}^{-\frac{l}{\varepsilon}} D_{\frac{\varepsilon}{l}}(), & < 1, \end{cases} \tag{16}$$

where $C_{\frac{\varepsilon}{l}} = G(0)$ and $D_{\frac{\varepsilon}{l}}() = G(u())$ with $G(a) = \int^{\infty} \frac{-v}{1+(\frac{\varepsilon}{l}-1)^{-1}} \frac{l}{\varepsilon} dv$, and $u() \equiv \frac{\varepsilon}{} - 1 \quad \frac{1}{} - 1$.

See Appendix F. ■

Fig. 4. (a) Comparing the SINR distributions for various fading distributions and noise profiles (Nakagami-2 refers to the Nakagami distribution with a shape parameter 2 and mean 23.45, Exp(23.45) refers to an exponential random variable with mean 23.45, logN(0,8 dB) refers to a log-normal random variable whose natural logarithm has a mean and variance of 0 and 8 dB, respectively). (b) Evaluating the tightness of the few-BS approximation.

in the region $\in [1, \infty)$, when the tail probability is plotted against , both in the logarithmic scale. This shows that the few BS approximation characterizes the signal quality in closed form and is a good approximation for the actual SIR.

Now, having characterized the SIR for the homogeneous l-D SCS, we look closely into what happens when $\leq l$. We will restrict ourselves to the case when $l = 2$, and the steps are similar for $l = 1$, and $l = 3$.

A homogeneous 2-D SCS with BS density λ, where the signal decays according to a power-law path-loss function with a path-loss exponent ≤ 2, the SIR at the MS is 0 with probability 1.

See Appendix G for the case $= 2$. From [17, Corollary 5], $\mathbb{P}(\{\mathrm{SIR} > \})|_{\varepsilon<2} \leq \mathbb{P}(\{\mathrm{SIR} > \})|_{\varepsilon=2} = 0, \forall \geq 0$. Hence we have proved the above result. ■

Note that once we have characterized the SINR distribution, the outage probability at the MS is known. The event that the MS is in coverage is given by $\{\mathrm{SINR} > \}$, where is the SINR threshold that the MS should satisfy to be in coverage. Consequently, the coverage probability, $\mathbb{P}(\{\mathrm{SINR} > \})$ is precisely the tail probability of SINR computed at . Next, we study the area-averaged spectral efficiency [33, Page 77] for an MS in coverage. This quantity, termed as the coverage conditional average rate, is given by $\mathcal{R} = \mathbb{E}[\log(1 + \mathrm{SINR})|\{\mathrm{SINR} > \}]$ and is the average of the instantaneous rate achievable at the MS when the interference is considered as noise. The coverage conditional average rate at the MS simplifies to the following expression.

$$\mathcal{R} = \log(1 +) + \int^{\infty} \frac{\mathbb{P}(\{\mathrm{SINR} > t\})}{(1+t)\mathbb{P}(\{\mathrm{SINR} > \})} dt.$$

As a result, based on Proposition 1 and Theorems 1-2, we can compute the coverage conditional average rate for any SCS. Specifically, in the interference-limited case, the following proposition provides the expression for a homogeneous l-D SCS and when the popular power-law path-loss model is assumed. For this case, the SIR characteristics are invariant to the randomness in the transmission powers and the fading factors due to Remark 2. Hence, without loss of generality, we restrict our attention to the case of constant transmission powers at all BSs and no fading.

2 The ergodic average rate at the MS in a homogeneous 2-D SCS under the power-law path-loss model, with constant transmission powers at all BSs and no fading is given by

$$\mathcal{R} = \log(1 +) + \int \frac{\mathbb{P}(\{\mathrm{SIR} > x\})}{\mathbb{P}(\{\mathrm{SIR} > \})(1+x)} dx + \ -\frac{2}{\varepsilon} \frac{}{2} \cdot {}_2F_1\left(1, \frac{2}{\varepsilon}; 1 + \frac{2}{\varepsilon}; - {}^{-1}\right),$$

where $= \max(\ , 1)$, where ${}_2F_1\left(1, \frac{2}{\varepsilon}; 1 + \frac{2}{\varepsilon}; - {}^{-1}\right)$ is the Gauss hypergeometric function and the probabilities are computed using (4). Note that for ≥ 1, the middle term drops out.

V. Numerical Example and Discussion

In the first example, we consider a homogeneous 2-D SCS with $\lambda = 0.01$, a power-law path-loss model with path-loss exponent 4, and a background noise power of -10 dB and unity transmission powers. We compare the SINR tail probabilities for several cases where we vary the distributions of the fading factors as well as the background noise power. Notice in Fig. 4(a) that in the case when there is background noise, the distribution of the fading greatly affects the SINR performance at the MS. We consider three examples for the i.i.d. fading factors: Nakagami distribution with a shape parameter 2, exponential distribution and log-normal distribution, and keep the same mean (=23.45) for all the cases, for a fair comparison. In the presence of the background noise, the MS sees a better SINR performance for the Nakagami and the exponential case compared to the log-normal case and the SINR performance in all cases is far more superior than that without fading. This is justified by Corollary 3 and Lemma 4 where the equivalent homogeneous 2-D SCS with unity BS density has an equivalent background noise power for the log-normal fading case that is strictly greater than that for the exponential fading and the Nakagami distributions. Further, in the no noise case, the SINR performance is invariant to the fading distribution and is the same as in the no fading case. This is also depicted in Fig. 4(a).

In Fig. 4(b), we assess the few-BS approximation for the SIR characterization in the homogeneous l-D SCS. This figure shows that the SIR approximation derived in Section IV based on the few-BS approximation (Equation (17)) closely follows the exact SIR characterization. Moreover, this relationship holds for a wide range of scenarios of interest such as for arbitrary fading and transmission power distributions, and for all BS densities. In the following section, we discuss the usage of the results obtained thus far in the analysis of other useful wireless communication scenarios.

VI. Applications in all4(a)-3 usage

and hence separates out as e^{ω} ^{1} from the original conditional characteristic function expression in (a); (c) is obtained by rewriting the exponential of summation in the characteristic function term in (b) as a product of exponentials; (d) is obtained by first noting that conditioned on R_1, the events in the two disjoint regions $[0, R_1]$ and (R_1, ∞) are independent of each other, and hence by the Restriction theorem [1, Page 17], all the points beyond R_1, represented by the set $\{R\}_{=1}^{\infty}$ can be regarded to be associated with a Poisson point process in 1-D restricted to the region (R_1, ∞), and with a density function $\lambda(r)$. As a result, now we can apply Campbell's theorem [1, (3.18), p. 28] to the inner expectation in (c) to obtain (d), which is further simplified to obtain (6).

. 2

Here, we derive the expression for the tail probability of SINR for values greater than or equal to 1. Due to [32, Lemma 1], there exists a unique BS within the 1-D SCS such that ≥ 1 holds true. Suppose the index of this unique BS is i. Then the expression for the tail probability of SINR, the SINR at the user when receiving from this BS, is given by

$$\mathbb{P}(\{\mathrm{SINR} > \})$$

$$\overset{(\,)}{=} \mathbb{P}\left(\frac{R^{-1}}{\sum_{=1, \neq}^{\infty} R^{-1} +} > \right)$$

$$\overset{(\,)}{=} \mathbb{E}\left[(- R) \prod_{=1, \neq} - R R^{-1}\right]$$

$$\overset{(\,)}{=} \mathbb{E}[(- R) \times - \int_{=0}^{\infty} \left(1 - \mathbb{E}\, ^{-1}\right) \lambda(r) dr$$

$$\overset{(\,)}{=} \mathbb{E}[(- R) \times - \int_{=0}^{\infty} \left(1 - \frac{1}{1 + R r^{-1}}\right) \lambda(r) dr,$$

where (a) is the expression for the tail probability of SINR of the 1-D SCS with BS density $\lambda(r)$ for which $\{R\}_{=1}^{\infty}$ is the set of distances of BSs from the MS and 'i' is the index of the unique BS that can satisfy the constraint $\{\mathrm{SINR} > \}$; (b) is obtained by evaluating the expectation w.r.t. and the expectation operator $\mathbb{E}$ is w.r.t. to all other random variables in (a); (c) is obtained by first conditioning w.r.t. R and by Slivnyak's theorem noting that the Palm distribution (see [21, Chapter 8] and [3, Chapter 13] for details on Palm theory and Slivynak's theorem) of the BSs represented by $\{R\}_{=1, \neq}^{\infty}$ given a BS at R is still a Poisson point process with density function $\lambda(r)$, then applying the Marking theorem [1, Page 55] and Campbell's theorem [1, (3.18), p. 28] where is the unity mean exponential random variable; (d) is obtained by evaluating the expectation in (c); and finally, since there is a unique BS i such that $\mathrm{SINR} \geq 1$, we can write the tail probability of SINR as $\mathbb{P}(\{\mathrm{SINR} > \}) = \mathbb{P}(\cup_{=1}^{\infty}\{\mathrm{SINR} > \}) = \sum_{=1}^{\infty} \mathbb{P}(\{\mathrm{SINR} > \})$

$\mathbb{E}\left[\prod_{=1}^{\infty} (- R) - \sum_{=0}^{\infty} \frac{(\,)}{1+(\ \ _i)^{-1}}\right]$ (from (d)) = (7) from Campbell's Theorem [1, (3.18), p. 28].

. (4)

First, using Corollary 5, $\mathrm{SIR}_2 = \frac{K\ _1^{-\varepsilon}}{P_I(2)}$, with $P_I(2) = K R_2^{-\varepsilon}\left(1 + \frac{0\ I}{\varepsilon -} R_2\right)$. Next, notice that the event $\{\mathrm{SIR}_2 > \}$ is equivalent to the joint event $\left(R_1 \leq R_2, R_1 < \left(\frac{P_I(2)}{K}\right)^{-\frac{1}{\varepsilon}}\right)$ and thus, $\mathbb{P}(\{\mathrm{SIR}_2 > \}) = \mathbb{P}\left(R_1 \leq \left(R_2, \left(\frac{P_I(2)}{K}\right)^{-\frac{1}{\varepsilon}}\right)\right)$, where

$$\left(R_2, \left(\frac{P_I(2)}{K}\right)^{-\frac{1}{\varepsilon}}\right)$$

$$= \begin{cases} \left(\frac{P_I(2)}{K}\right)^{-\frac{1}{\varepsilon}}, & \geq 1 \\ \left(\frac{P_I(2)}{K}\right)^{-\frac{1}{\varepsilon}}, & < 1, R_2 > \left(\frac{\times (\,)}{0\ I}\right)^{\frac{1}{I}} \\ R_2, & < 1, R_2 \leq \left(\frac{\times (\,)}{0\ I}\right)^{\frac{1}{I}}. \end{cases}$$

Finally, (16) is obtained using the joint p.d.f., $f_{1,2}(r_1, r_2) = (\lambda_0 b)^2 (r_1 r_2)^{-1} - \frac{0\ I}{} r_2$, $0 \leq r_1 \leq r_2 \leq \infty$, due to the properties of Poisson point processes.

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Let us consider the probability of the event that the interference due to all the BSs beyond the signal BS at a given distance R_1 is below a certain value, say, , for the case $= 2$.

$$\mathbb{P}\left(\sum_{=2} R^{-2} \leq \,\middle|\, R_1\right)$$

$$= \mathbb{P}\left(- \sum_{k=2}^{\infty} {}_k^{-2} \geq - \,\middle|\, R_1\right)$$

$$\overset{(\,)}{\leq} \mathbb{E}\left[- \sum_{k=2}^{\infty} {}_k^{-2} \,\middle|\, R_1\right]$$

$$\overset{(\,)}{=} - \int_{r=R_1}^{\infty} \left(1 - \ ^{-s r^{-2}}\right))$$

As a result,

$$\mathbb{P}(\{SIR > \}) = \mathbb{E}_{1}\, \mathbb{P} \sum_{=2} R^{-\varepsilon} < (R_1^{\varepsilon})^{-1} \Big| R_1$$

$$= 0, \forall \geq 0.$$

and hence we have proved the result.

[illegible]

In this section, the details of simulating the SCS are presented. A single trial in simulating the BS placement for the 1-D SCS with BS density function $\lambda(r)$ in the region of interest which is a subset of the 1-D plane denoted by S, involves the

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