A network-specific approach to percolation in complex networks with bidirectional links

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A network-specific approach to percolation in complex networks with bidirectional links

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Introduction. – The study of percolation in complex networks has broad applications including epidemic spreading [1], propagation of excitation in neural networks [2], and robustness of networks to random failure [3] or strategic attack [4,5]. A central problem is estimating the percolation threshold, the critical fraction of nodes or links of an initially connected network that must be removed to disintegrate it into small disconnected fragments. Knowledge of how a network fragments can improve strategies for designing attack [4,5] and immunization techniques [6] or increasing network robustness [7,8].

Several studies have proposed techniques to estimate the percolation threshold of a network for various situations [4,9–13]. These studies typically use estimated in important applications (e.g., the power grid [19] and air transportation networks [20]). Besides relaxing the ensemble assumptions of previous research (e.g., that the network is strictly Markovian), one significant advantage of this approach is that it can easily account for arbitrary strategies of node/link removal. Network-specific approaches are therefore well suited for developing network-specific attack/defense strategies, immunization techniques, etc. In addition to estimating the percolation threshold, we predict the expected number of nodes accessible to each node after the network disintegrates. This has various applications such as predicting the outbreak size of an epidemic [1]. We finally show that our method may be used to study the fragmentation of a network subject to either probabilistic or deterministic attack.

Anal i . We formalize weighted percolation (i.e., in which nodes and/or links are retained with arbitrary probabilities) as follows: for a network with N nodes described by a possibly asymmetric adjacency matrix A ($A_{nm} = 1$, if a link exists from node n to node m and A_{nm}

requires the invertibility of the matrix $I - D(q)$. This matrix is invertible when $\lambda_{D(q)} < 1$, where $\lambda_{D(q)}$ is the principal eigenvalue of $D(q)$. As $\lambda_{D(q)} \to 1^-$ the out-component sizes diverge as $s_n^{out} \sim [1 - \lambda_{D(q)}]^{-1} w_n$, where is the principal eigenvector of $D(q)$. A similar argument can be made for the divergence of the in-component sizes. Since the LSCC above the percolation threshold can be thought of as the set of vertices with infinite in- and out-components [11], we predict the percolation threshold as

$$q_D^* = \min_{q \in [0,1]} \{q : \lambda_{D(q)} = 1\}. \quad (6)$$

We note that if there are no bidirectional links, $\hat{A}_{nm}\hat{A}_{mn} = 0$ and $D(q) = q\hat{A}$, and the results of ref. [13] are recovered. While one may solve eq. (6) numerically, it is both practical and insightful to approximate eqs. (2)–(6) for large s_n^{out} and small q. Letting $s_n^{out} \gg 1$ and $\beta_{nm} \sim 1$ in eq. (1) yields the approximate eigenvalue problem $s_n^{out} \approx q \sum_m \hat{A}_{nm} s_m^{out}$. It follows that $^{out} \propto$. Upon substitution we find $q \sim \hat{\lambda}^{-1}$ under these conditions, yielding to first order $\beta_{nm} \approx 1 - \hat{\lambda}^{-1}\hat{A}_{mn}\hat{u}_n/\hat{u}_m$. Defining

$$C_{nm} = \hat{A}_{nm}\left(1 - \frac{\hat{A}_{mn}\hat{u}_n}{\hat{\lambda}\hat{u}_m}\right), \quad (7)$$

with principal eigenvalue equation $C = \lambda_C$ and using $\approx \mathbf{1} = [1, 1, \ldots, 1]^T$, we obtain the predictions

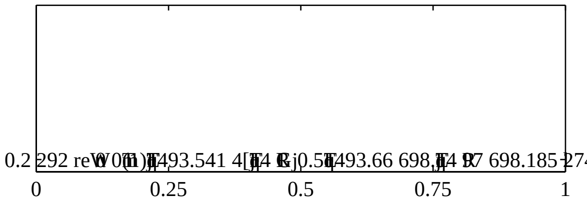
$$^{out} \approx (I - qC)^{-1}\mathbf{1}, \quad (8)$$

$$q_C^* \approx \lambda_C^{-1}. \quad (9)$$

In addition to o ering simplified predictions for s_n^{out} and q^*, for unweighted percolation (i.e., $\hat{A} = A$ and $=$) these estimates allow us to bound λ_C using the principal eigenvalue λ of the network adjacency matrix A (e.g., $A = \lambda$). Direct application of the Bauer-Fike theorem [22] for the limiting case of an undirected network yields $|\lambda_C - \lambda| \leqslant ||\lambda^{-1}UAU^{-1}||_2 = 1$, where $U = \mathrm{diag}[u_1, \ldots, u_N]$. Finally, considering $\mathbf{1}^T C$ and using $\sim$ yields $\lambda_C \approx \lambda - 1$. One implication of these results is that $q^* \to 0$ for large λ, which is consistent with the lack of an unweighted percolation threshold for well-connected networks such as scale-free networks [9]. We note that eqs. (8) and (9) are in best agreement with eqs. (3) and (6) near $q = q^*$ and when the network is strictly undirected or strictly directed.

E ample . In what follows, we will motivate the need for our theory, recover previous results, explore several applications, and illustrate the robustness of our analysis to complex structures in networks. We will consider both computer-generated and real-world networks.

We first highlight the need for a network-specific method for undirected networks and show that unlike the ensemble approach, a network-specific method captures variability in q^*



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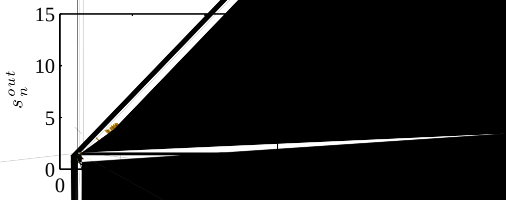
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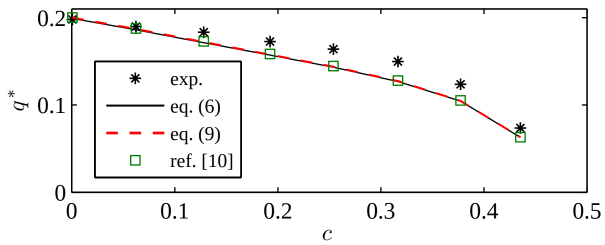

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Concl ion . We have presented a network-specific approach to weighted percolation in undirected networks and directed networks with bidirectional links (previously open problems). As opposed to most previous theory dealing with network ensembles, our .546 726.7rks and d..6278384- 0 -hileen problems)..546 72-3755etwor]TJ etbles, o -188(As)-2cificwnetwta.4(wi2004 s.000.1(rategies(wi2