

impounded dynamical processes on complex networks are defined on this fact, the dependence of the relative change in the largest eigenvalue of the network adjacency matrix on the relative change in the largest eigenvalue of the network adjacency matrix can be affected by the degree of the node and the network connectivity. The relative change in the largest eigenvalue of the network adjacency matrix can be expressed in terms of the relative change in the largest eigenvalue of the network adjacency matrix.

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In recent years, there has been much interest in the study of the structure of networks arising from real world systems, of dynamical processes taking place on networks, and of how network structure impacts such dynamics [1]. The largest eigenvalue of the network adjacency matrix (which we denote λ) is the key quantity determining a variety of different dynamical processes on networks. For example, (i) for a heterogeneous collection of chaotic and/or periodic dynamical systems coupled by a network of connections, the critical coupling strength [2] for the emergence of coherence is proportional to $1/\lambda$; (ii) in a class of percolation problems on directed networks [closely related to the problem of epidemic spreading [3]], the condition for the emergence of a giant component also involves λ [4]. For other examples where λ plays a similar role, see Refs. [5–7].

In many situations it might be desirable to control dynamical processes that take place on networks. For example, in epidemic spreading, one would like to increase the threshold for epidemic transmission. In percolation, one might like to identify the key nodes holding the network together and protect them (e.g., in the transportation network or the Internet) or disrupt them (e.g., in the case of a terrorist network or pathogen protein network). Such strategies would greatly benefit from a quantitative characterization of the effect of the removal of the different nodes or edges in the network. We will define the *dynamical importance* of nodes and edges as the relative change in the largest eigenvalue of the network adjacency matrix upon their removal. This provides an objective quantification of the relative importance of the different elements of the network that could potentially be used to formulate control strategies for those network processes that are governed by the largest eigenvalue of the network adjacency matrix. We also will describe an efficient way to approximate the dynamical importance.

We consider a network as a directed graph with N nodes, and we associate to it a $N \times N$ adjacency matrix whose elements A_{ij} are positive if there is a link going from node i to node j with $i \neq j$ and zero otherwise ($A_{ii} \equiv 0$). We denote the largest eigenvalue of A by λ , where $\lambda = \lambda_{\max}(A)$ and \mathbf{v} and \mathbf{w} with $\mathbf{w}^T A = \lambda \mathbf{w}^T$ and $A \mathbf{v} = \lambda \mathbf{v}$ denoting the right and left eigenvectors of A . According to Perron's theorem [7], of all the eigenvalues of A , the one with largest magnitude is real and positive, and the corresponding right and left eigenvectors are positive. The relative change in the largest eigenvalue of A upon the removal of a node i is given by $\Delta \lambda_i / \lambda$.

$$(1 + \epsilon)(1 + \delta) = (1 + \delta)(1 + \epsilon) \quad (3)$$

by neglecting second order terms and $\epsilon, \delta \ll 1$, we obtain $\epsilon = -\delta$. Upon removal of edge (i, j) , the perturbation matrix is $(\epsilon)_{ij} = -\delta_{ij}$, and therefore

$$\hat{A} = A - \delta_{ij} e_i e_j^T \quad (4)$$

We now examine the effect of removing node i . Upon its removal, the perturbation matrix is given by $(\epsilon)_{ij} = -\delta_{ij}$. However, in this case we cannot assume ϵ is small as we did before, since $\epsilon_{ii} = -1$ (the left and right eigenvectors have zero i -th entry after the removal of node i). Therefore, we set $\epsilon = -\delta e_i e_i^T$, where e_i is the unit vector for the i -th component, and we assume $\delta \ll 1$.

Our next example is motivated by the fact that it is sometimes observed that real networks can be subdivided into more or less well defined communities which have different statistics, and thus potentially different dynamical importance. As a simple model of such situation, we specify a division of the nodes in the network into two groups of the same size, n_A and n_B , ($n_A \cup n_B = \{A$

where n is the number of removed nodes and λ_1 is the largest eigenvalue of the resulting network. We see that using the dynamical importance (solid lines) greatly improves the results over using the degree (short dashed